PURE 3 JAN 2003 Q4

i) $l_1: \underline{r} = \begin{pmatrix}1\\8\\8\end{pmatrix} + \lambda\begin{pmatrix}0\\3\\4\end{pmatrix}$

$l_2: \underline{r} = \begin{pmatrix}3\\3\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$

$P(1, 2, 0)$

$l_1 \quad \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}1\\8\\8\end{pmatrix} - 2\begin{pmatrix}0\\3\\4\end{pmatrix}$

$\lambda = -2$, P is on l_1

$l_2 \quad \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}3\\3\\-1\end{pmatrix} - 1\begin{pmatrix}2\\1\\-1\end{pmatrix}$

$\mu = -1$, P is on l_2

ii)

$$\cos\theta = \frac{\begin{pmatrix}0\\3\\4\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}}{\left|\begin{pmatrix}0\\3\\4\end{pmatrix}\right|\left|\begin{pmatrix}2\\1\\-1\end{pmatrix}\right|}$$

$$\cos\theta = \frac{0+3-4}{\sqrt{25}\sqrt{6}}$$

$$\cos\theta = \frac{-1}{5\sqrt{6}}$$

$$\theta = 94.7^\circ$$

Acute angle $= 180 - 94.7$

$= 85.3^\circ$

iii) $2x - y + z = 12$

A Normal $\begin{pmatrix}2\\-1\\1\end{pmatrix}$

Line through P $\perp$ to plane

$\underline{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$

iv) Point on line $= \begin{pmatrix}1+2\lambda\\2-\lambda\\\lambda\end{pmatrix}$

Subst in plane

$2(1+2\lambda) - (2-\lambda) + \lambda = 12$

$2 + 4\lambda - 2 + \lambda + \lambda = 12$

$6\lambda = 12$

$\lambda = 2$

Line meets plane at

$(1+2(2), 2-2, 2)$

$= (5, 0, 2)$

Distance of P from plane

$= \sqrt{(5-1)^2 + (0-2)^2 + (2-0)^2}$

$= \sqrt{16+4+4}$

$= \sqrt{24}$

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PURE 3 JUNE 2003 Q4

i) A(2, 0, 2)
B(-2, 0, 1)
C(0, 4, 3)

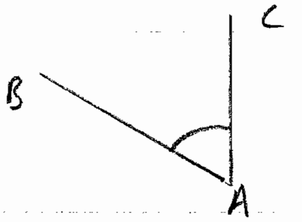
i) $|\overrightarrow{AC}| = \sqrt{(2-0)^2+(0-4)^2+(2-3)^2}$

$= \sqrt{4+16+1}$

$= \sqrt{21}$

ii) $\overrightarrow{AB}.\overrightarrow{AC} = \begin{pmatrix}-4\\0\\-1\end{pmatrix}.\begin{pmatrix}-2\\4\\1\end{pmatrix}$

$= 8+0-1 = 7$



$\cos\angle BAC = \dfrac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$

$|\overrightarrow{AB}| = \sqrt{(2--2)^2+0^2+1^2}$

$= \sqrt{17}$

$\cos\angle BAC = \dfrac{7}{\sqrt{21}\sqrt{17}}$

$\angle BAC = 68.3°$

iii) $\begin{pmatrix}2\\3\\-8\end{pmatrix}.\begin{pmatrix}-4\\0\\-1\end{pmatrix} = -8+0+8 = 0$

$\therefore \begin{pmatrix}2\\3\\-8\end{pmatrix}$ and AB are $\perp$

$\begin{pmatrix}2\\3\\-8\end{pmatrix}.\begin{pmatrix}-2\\4\\1\end{pmatrix} = -4+12-8=0$

$\therefore \begin{pmatrix}2\\3\\-8\end{pmatrix}$ and AC are $\perp$

Since $\begin{pmatrix}2\\3\\-8\end{pmatrix}$ is $\perp$ to 2 non parallel lines in the plane, it is $\perp$ to the plane.

Plane is of form

$2x+3y-8z=d$

Subst A(2, 0, 2) in plane

$2(2)+3(0)-8(2)=d$
$4+0-16=d$
$\Rightarrow d=-12$

Plane is $2x+3y-8z=-12$

iv) H above G so H is $(1, 1, k)$ say.
Subst H in plane

$2(1)+3(1)-8(k)=-12$
$2+3-8k=-12$
$17=8k$
$\Rightarrow k=\dfrac{17}{8}$

k is height of pole

so pole is $\dfrac{17}{8}$ metres

PURE 3 JAN 2004 Q4

i) A(1, 0, 0)
B(0, 1, 0)
C(0, 0, 2)
D(−2, 0, 0)

$$\vec{BA}\cdot\vec{BC} = \begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\2\end{pmatrix}$$

$$= 0 + 1 + 0 = 1$$

$$\cos\angle ABC = \frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}||\vec{BC}|}$$

$$\cos\angle ABC = \frac{1}{\sqrt{2}\sqrt{5}}$$

$$\angle ABC = 71.6^\circ$$

ii) Verify A, B and C on $2x + 2y + z = 2$

A
$$2(1) + 2(0) + (0) = 2$$
$$2 + 0 + 0 = 2 \checkmark$$

B
$$2(0) + 2(1) + (0) = 2$$
$$0 + 2 + 0 = 2 \checkmark$$

C
$$2(0) + 2(0) + (2) = 2$$
$$0 + 0 + 2 = 2 \checkmark$$

∴ plane ABC is $2x + 2y + z = 2$

A normal to plane is $\begin{pmatrix}2\\2\\1\end{pmatrix}$

iii) A ⊥ line from D(−2,0,0) to plane ABC

$$\underline{r} = \begin{pmatrix}-2\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$$

iv) A point on line

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2+2\lambda\\2\lambda\\\lambda\end{pmatrix}$$

Subst in plane

$$2(-2+2\lambda) + 2(2\lambda) + (\lambda) = 2$$
$$-4 + 4\lambda + 4\lambda + \lambda = 2$$
$$9\lambda = 6$$
$$\lambda = \frac{2}{3}$$

Point of intersection with plane

$$= \left(-2+2\left(\tfrac{2}{3}\right), 2\left(\tfrac{2}{3}\right), \left(\tfrac{2}{3}\right)\right)$$
$$= \left(-\tfrac{2}{3}, \tfrac{4}{3}, \tfrac{2}{3}\right)$$

Distance of D from plane

$$= \sqrt{\left(-2 - -\tfrac{2}{3}\right)^2 + \left(-\tfrac{4}{3}\right)^2 + \left(-\tfrac{2}{3}\right)^2}$$
$$= \sqrt{\tfrac{16}{9} + \tfrac{16}{9} + \tfrac{4}{9}}$$
$$= \sqrt{\tfrac{36}{9}}$$
$$= \sqrt{4} = 2 \text{ units}$$

PURE 3 JUNE 2004 Q3

$l_1 \quad \underline{r} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\2\end{pmatrix}$

$l_2 \quad \underline{r} = \begin{pmatrix}0\\2\\2\end{pmatrix} + \mu\begin{pmatrix}-1\\1\\0\end{pmatrix}$

i) A has position vector

$$\underline{r} = \begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}2\\1\\4\end{pmatrix}$$

B has position vector

$$\underline{r} = \begin{pmatrix}0\\2\\2\end{pmatrix} + 3\begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}-3\\5\\2\end{pmatrix}$$

$$|AB| = \sqrt{(2--3)^2 + (1-5)^2 + (4-2)^2}$$
$$= \sqrt{25 + 16 + 4}$$
$$= \sqrt{45}$$

ii)

$$\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$\lambda = 1$, $(1,1,2)$ on l_1

$$\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}0\\2\\2\end{pmatrix} - 1\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$\mu = -1$, $(1,1,2)$ on l_2

$$\cos\angle ACB = \frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{|\overrightarrow{CA}|\,|\overrightarrow{CB}|}$$

$\overrightarrow{CA} = \begin{pmatrix}1\\0\\2\end{pmatrix} \qquad \overrightarrow{CB} = \begin{pmatrix}-4\\4\\0\end{pmatrix}$

$$\cos\angle ACB = \frac{\begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}-4\\4\\0\end{pmatrix}}{\sqrt{5}\,\sqrt{32}}$$

$$= \frac{-4+0+0}{\sqrt{5}\,\sqrt{32}}$$

$$\angle ACB = 108.4^\circ$$

iii) $\begin{pmatrix}2\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix} = 2+0-2 = 0$

$\therefore \begin{pmatrix}2\\2\\-1\end{pmatrix}$ is $\perp$ to l_1

$\begin{pmatrix}2\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix} = -2+2+0 = 0$

$\therefore \begin{pmatrix}2\\2\\-1\end{pmatrix}$ is $\perp$ to l_2

Since l_1 and l_2 are non-parallel $\underline{n} = \begin{pmatrix}2\\2\\-1\end{pmatrix}$ is $\perp$ to plane containing l_1 and l_2

Plane of form $2x + 2y - z = d$

Subst $C(1,1,2)$ in plane

$$2(1) + 2(1) - (2) = d$$
$$2 + 2 - 2 = d$$
$$\Rightarrow d = 2$$

Plane is

$$2x + 2y - z = 2$$

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PURE 3 JAN 2005 Q4

i) $\overrightarrow{BC} = \begin{pmatrix} 1.4 \\ -0.2 \\ -1 \end{pmatrix}$ $\overrightarrow{BF} = \begin{pmatrix} 0.2 \\ 1.4 \\ -1 \end{pmatrix}$

ii) $|\overrightarrow{BC}| = \sqrt{1.4^2 + (-0.2)^2 + (-1)^2}$

$= \sqrt{3}$

$|\overrightarrow{BF}| = \sqrt{0.2^2 + 1.4^2 + (-1)^2}$

$= \sqrt{3}$

$\therefore \triangle BCF$ is isosceles

iii) $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1.4 \\ -0.2 \\ -1 \end{pmatrix} = 5.6 - 0.6 - 5 = 0$

$\therefore \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$ is $\perp$ to $\overrightarrow{BC}$

$\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 0.2 \\ 1.4 \\ -1 \end{pmatrix} = 0.8 + 4.2 - 5 = 0$

$\therefore \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$ is $\perp$ to $\overrightarrow{BF}$

Since it is $\perp$ to 2 non-parallel lines in plane, $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$ is normal to plane.

Plane is of form

$4x + 3y + 5z = d$

Subst $B(4,3,2)$ in plane.

$4(4) + 3(3) + 5(2) = d$
$16 + 9 + 10 = d$
$\Rightarrow d = 35$

Plane BCF is

$4x + 3y + 5z = 35$

iv) $3x - 4y + 5z = 10$

$A(0,0,2)$
$3(0) - 4(0) + 5(2) = 10$
$0 - 0 + 10 = 10$ ✓

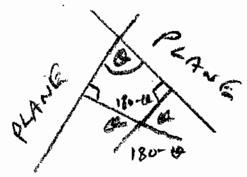
$B(4,3,2)$
$3(4) - 4(3) + 5(2) = 10$
$12 - 12 + 10 = 10$ ✓

$D(6,2,0)$
$3(6) - 4(2) + 5(0) = 10$
$18 - 8 + 0 = 10$ ✓

$\therefore$ plane $3x - 4y + 5z = 10$ passes through A, B and D

Normal $\underline{n}_2 = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$

v)



Angles between planes = Angles between normals

Angle between normals

$$\cos\theta = \frac{\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}}{\sqrt{16+9+25}\,\sqrt{9+16+25}}$$

$$= \frac{12 - 12 + 25}{\sqrt{50}\,\sqrt{50}}$$

$$\cos\theta = \frac{25}{50} = \frac{1}{2}$$

$\theta = 60°$

Acute angle between planes

$= 60°$

PURE 3 JUNE 2005 Q4

i) $\underline{a} = \begin{pmatrix} -3 \\ 4 \\ 12 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$

$\cos \angle AOB = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$

$= \dfrac{-6 + 16 + 48}{\sqrt{9+16+144}\sqrt{4+16+16}}$

$\cos \angle AOB = \dfrac{58}{\sqrt{169}\sqrt{36}} = \dfrac{58}{13 \times 6}$

$\angle AOB = 42.0°$

Area of $\Delta = \frac{1}{2}|\underline{a}||\underline{b}| \sin \angle AOB$

$= \frac{1}{2} \times 13 \times 6 \times \sin 42°$

$= 26.1$ units^2

ii) Line AD given by

$\underline{r} = \underline{a} + \lambda \underline{c}$

$\underline{r} = \begin{pmatrix} -3 \\ 4 \\ 12 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -9 \\ 5 \end{pmatrix}$

iii) $\underline{c} \cdot \underline{a} = \begin{pmatrix} 8 \\ -9 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \\ 12 \end{pmatrix} = -24 - 36 + 60 = 0$

$\underline{c} \cdot \underline{b} = \begin{pmatrix} 8 \\ -9 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} = 16 - 36 + 20 = 0$

$\underline{c}$ is $\perp$ to both $\underline{a}$ and $\underline{b}$

and is $\therefore$ $\perp$ to plane AOB

Planes OAB and CDE are of form

$8x - 9y + 5z = d$

Subst O(0, 0, 0) in to plane

$8(0) - 9(0) - 5(0) = d$

$\Rightarrow d = 0$

Plane OAB is given by

$8x - 9y + 5z = 0$

Subst C(8, −9, 5) into plane

$8(8) - 9(-9) + 5(5) = d$

$64 + 81 + 25 = d$

$\Rightarrow d = 170$

Plane CDE is given by

$8x - 9y + 5z = 170$

iv) Volume of prism

= Area of cross-section × length

$= 26.1 \times |\vec{OC}|$

$= 26.1 \times \sqrt{8^2 + (-9)^2 + 5^2}$

$= 26.1 \times \sqrt{170}$

$= 340.3$ units^3