MEI CORE 3 JAN 2009

1) $|x-1| < 3$

$-3 < x-1 < 3$

$-3+1 < x < 3+1$

$-2 < x < 4$

2) i) $\frac{d}{dx}\, x\cos 2x$

$= x(-2\sin 2x) + \cos 2x \times 1$

$= \cos 2x - 2x\sin 2x$

ii $\int x\cos 2x\,dx$

Let $u = x$ $\Rightarrow \frac{du}{dx} = 1$ Let $\frac{dv}{dx} = \cos 2x$ $\Rightarrow v = \frac{1}{2}\sin 2x$

$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx$

$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$

3) $f(x) = \frac{1}{2}\ln(x-1)$

$g(x) = 1 + e^{2x}$

Let $y = f(x) = \frac{1}{2}\ln(x-1)$

swap variables

$x = \frac{1}{2}\ln(y-1)$

$2x = \ln(y-1)$

$e^{2x} = y-1$

$1+e^{2x} = y$

$\therefore f^{-1}(x) = 1+e^{2x} = g(x)$

4) $\int_0^2 \sqrt{1+4x}\,dx$

Let $u = 1+4x$

$\frac{du}{dx} = 4$

$du = 4dx$

$\frac{1}{4}du = dx$

when $x = 2$, $u = 9$
$x = 0$, $u = 1$

Integral becomes

$\int_1^9 \frac{1}{4}u^{\frac{1}{2}}\,du$

$= \frac{1}{4}\left[\frac{u^{3/2}}{3/2}\right]_1^9$

$= \frac{1}{4}\left[\frac{2}{3}u^{3/2}\right]_1^9$

$= \frac{1}{4}\times\frac{2}{3}\left[9^{3/2} - 1^{3/2}\right]$

$= \frac{1}{6}\left[27 - 1\right] = \frac{26}{6}$

$= \frac{13}{3}$

MEI CORE 3 JAN 2009

5) $f(x) = 1 + \cos 2x$

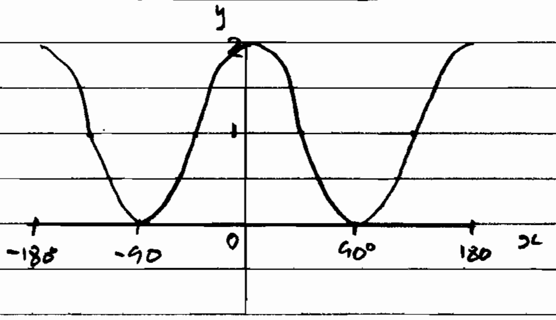
i) Period $= 180^\circ$

ii) Stretch parallel to x axis by scale factor $\frac{1}{2}$

Translation by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

These transformations could be applied in any order

iii)



6)

i) Disprove $p > q \Rightarrow \frac{1}{p} < \frac{1}{q}$

Let $p = +3$, $q = -4$

Then $p > q$

but $+\frac{1}{3} > -\frac{1}{4}$

$\therefore \frac{1}{p} \not< \frac{1}{q}$

Counter example disproves the proposition

ii) True if $p, q > 0$

ie (both p and $q > 0$)

7) $x^{2/3} + y^{2/3} = 5$

i) Differentiating gives

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\Rightarrow x^{-\frac{1}{3}} + y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\Rightarrow y^{-\frac{1}{3}}\frac{dy}{dx} = -x^{-\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{-x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}$$

$$\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{1/3}}$$

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

7ii)

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

when $x = 1$, $y = 8$ and $\frac{dx}{dt} = 6$

$$\frac{dy}{dt} = -\left(\frac{8}{1}\right)^{\frac{1}{3}} \times 6$$

$$\frac{dy}{dt} = -12$$

MEI CORE 3 JAN 2009

Section B

8) $y = x^2 - \frac{1}{8}\ln x$

i)

At P, $y = 1^2 - \frac{1}{8}\ln 1 = 1$

P is point $(1,1)$

R is point $(0, -\frac{7}{8})$

Gradient of PR $= \frac{1 - -\frac{7}{8}}{1-0}$

$= \frac{15}{8}$

ii)

$\frac{dy}{dx} = 2x - \frac{1}{8x}$

When $x = 1$, $\frac{dy}{dx} = 2 - \frac{1}{8} = \frac{15}{8}$

$\therefore$ PR is a tangent to curve at point P since it has the same gradient as the curve

iii)

At st. pt. $\frac{dy}{dx} = 0$

$\Rightarrow 2x - \frac{1}{8x} = 0$

$\Rightarrow 16x^2 - 1 = 0$

$\Rightarrow x^2 = \frac{1}{16}$

$\Rightarrow x = +\frac{1}{4}$

Since curve defined only for $x > 0$

When $x = \frac{1}{4}$, $y = \left(\frac{1}{4}\right)^2 - \frac{1}{8}\ln\frac{1}{4}$

$y = \frac{1}{16} - \frac{1}{8}\ln\frac{1}{4}$

or $y = \frac{1}{16} + \frac{1}{8}\ln 4$

$Q\left(\frac{1}{4}, \frac{1}{16} + \frac{1}{8}\ln 4\right)$

$\left(\text{or } Q\left(\frac{1}{4}, \frac{1}{16} - \frac{1}{8}\ln\frac{1}{4}\right)\right)$

iv)

$\frac{d}{dx}\, x\ln x - x$

$= x\frac{1}{x} + \ln x \times 1 - 1$

$= 1 + \ln x - 1$

$= \ln x$

Area $= \int_1^2 \left(x^2 - \frac{1}{8}\ln x\right)dx$

$= \left[\frac{x^3}{3} - \frac{1}{8}(x\ln x - x)\right]_1^2$

$= \left[\frac{x^3}{3} - \frac{1}{8}x(\ln x - 1)\right]_1^2$

$= \left(\frac{8}{3} - \frac{2}{8}(\ln 2 - 1)\right) - \left(\frac{1}{3} - \frac{1}{8}(0-1)\right)$

$= \frac{8}{3} - \frac{2}{8}\ln 2 + \frac{2}{8} - \frac{1}{3} - \frac{1}{8}$

$= \frac{7}{3} - \frac{1}{4}\ln 2 + \frac{1}{8}$

$= \frac{56}{24} - \frac{1}{4}\ln 2 + \frac{3}{24} = \frac{59}{24} - \frac{1}{4}\ln 2$

9) $f(x) = \dfrac{1}{\sqrt{2x-x^2}}$

i) asymptote when $2x - x^2 = 0$

$x(2-x) = 0$

$\Rightarrow x = 0$ or $x = 2$

$\therefore a = 2$

ii)

$y = (2x-x^2)^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = -\frac{1}{2}(2x-x^2)^{-\frac{3}{2}}(2-2x)$

$= -(2x-x^2)^{-3/2}(1-x)$

$= \dfrac{x-1}{(2x-x^2)^{3/2}}$

At turning point $\dfrac{dy}{dx} = 0$

$\Rightarrow x - 1 = 0$

$x = 1$

when $x = 1$, $y = \dfrac{1}{\sqrt{2\times1-1^2}} = 1$

Turning point is $(1,1)$

Range of function is

given by $f(x) \geq 1$

iii)

$g(x) = \dfrac{1}{\sqrt{1-x^2}}$

A) $g(-x) = \dfrac{1}{\sqrt{1-(-x)^2}}$

$= \dfrac{1}{\sqrt{1-x^2}} = g(x)$

Since $g(-x) = g(x)$

$g(x)$ is an even function

B) $g(x-1) = \dfrac{1}{\sqrt{1-(x-1)^2}}$

$= \dfrac{1}{\sqrt{1-(x^2-2x+1)}}$

$= \dfrac{1}{\sqrt{2x-x^2}} = f(x)$

C) $g(x)$ is an even function and so is symmetrical about y axis.

Since $f(x)$ is obtained from $g(x)$ by a translation of $\begin{pmatrix}1\\0\end{pmatrix}$, $f(x)$ is also symmetrical and has a line of symmetry

$x = 1$

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