OCR
RECOGNISING ACHIEVEMENT

General Certificate of Education
Advanced Supplementary (AS) and Advanced Level
former Oxford and Cambridge modular syllabus

MEI STRUCTURED MATHEMATICS 5505

Pure Mathematics 5

Tuesday **5 JUNE 2001** Morning 1 hour 20 minutes

Additional materials:
Answer paper
Graph paper
Students' Handbook

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your name, Centre number and candidate number in the spaces provided on the answer paper/ answer booklet.

Answer any **three** questions.

Write your answers on the separate answer paper provided.

If you use more than one sheet of paper, fasten the sheets together.

INFORMATION FOR CANDIDATES

The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.

You are advised that an answer may receive no marks unless sufficient detail of the working is shown on the answer paper to indicate that a correct method is being used.

This question paper consists of 3 printed pages and 1 blank page.

1 The cubic equation $2x^3+8x^2+13x-10=0$ has roots α, β and γ.

(i) Write down the values of $\alpha+\beta+\gamma$, $\beta\gamma+\gamma\alpha+\alpha\beta$ and $\alpha\beta\gamma$. [3]

(ii) Find $\alpha^2+\beta^2+\gamma^2$. [2]

(iii) By considering $(\alpha+\beta+\gamma)(\beta\gamma+\gamma\alpha+\alpha\beta)$, or otherwise, show that

$$\alpha^2\beta+\alpha^2\gamma+\beta^2\gamma+\beta^2\alpha+\gamma^2\alpha+\gamma^2\beta=-41.$$ [3]

(iv) Find $\alpha^3+\beta^3+\gamma^3$. [4]

(v) Find a cubic equation with integer coefficients which has roots

$$\frac{\beta+\gamma}{\alpha},\quad \frac{\gamma+\alpha}{\beta} \quad\text{and}\quad \frac{\alpha+\beta}{\gamma}.$$ [8]

2 (a) By considering $(\cos\theta+\mathrm{j}\sin\theta)^4$, express $\tan 4\theta$ in terms of $\tan\theta$. [5]

(b) (i) By considering $\left(z-\frac{1}{z}\right)^2\left(z+\frac{1}{z}\right)^4$, where $z=\cos\theta+\mathrm{j}\sin\theta$, show that

$$\sin^2\theta\cos^4\theta=\tfrac{1}{16}+\tfrac{1}{32}\cos 2\theta-\tfrac{1}{16}\cos 4\theta-\tfrac{1}{32}\cos 6\theta.$$ [8]

(ii) Use the substitution $x=\tan\theta$ to show that

$$\int_0^1 \frac{x^2}{\left(1+x^2\right)^4}\,\mathrm{d}x=\frac{\pi}{64}+\frac{1}{48}.$$ [7]

3 (i) Given that $k \geqslant 1$ and $\cosh x=k$, prove that $x=\pm\ln\left(k+\sqrt{k^2-1}\right)$. [5]

In the remainder of this question, $\mathrm{f}(x)=2\sinh^2 x-5\cosh x$.

(ii) Solve the equation $\mathrm{f}(x)=10$, giving your answers in an exact logarithmic form. [4]

(iii) Find the coordinates of the stationary points on the curve $y=\mathrm{f}(x)$. [6]

(iv) Show that $\displaystyle\int_0^{\ln 10} \mathrm{f}(x)\,\mathrm{d}x=\frac{99}{400}-\ln 10$. [5]

4 $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are distinct points on the rectangular hyperbola $xy = c^2$.

The tangent at P meets the x-axis at A and the y-axis at B. The tangent at Q meets the x-axis at C and the y-axis at D.

(i) Find the equation of the tangent at P. [4]

(ii) Show that P is the midpoint of AB. [3]

(iii) Show that the three lines AD, CB and PQ are parallel. [4]

The foci of the hyperbola are $F(c\sqrt{2}, c\sqrt{2})$ and $G(-c\sqrt{2}, -c\sqrt{2})$.

(iv) Sketch the hyperbola, and show the positions of F and G.

Using standard focal distance and reflection properties of the hyperbola, or otherwise, find

(*A*) $|PF - PG|$,

(*B*) the gradient of the bisector of angle FPG,

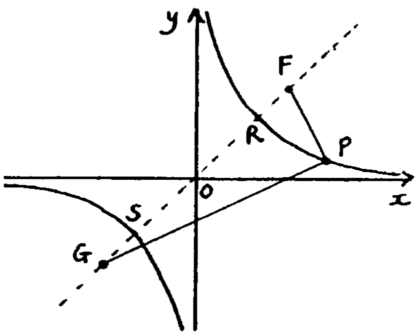
giving your answers as simply as possible. [9]

Mark Scheme

1 (i)	$\sum\alpha=-4,\ \sum\alpha\beta=\frac{13}{2},\ \alpha\beta\gamma=5$	B1B1B1 3	
(ii)	$\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta=(-4)^2-2(\frac{13}{2})$ $=3$	M1 A1 cao 2	Correct formula and substituting numbers
(iii)	$\left(\sum\alpha\right)\left(\sum\alpha\beta\right)=\sum\alpha^2\beta+3\alpha\beta\gamma$ $\sum\alpha^2\beta=(-4)(\frac{13}{2})-3(5)=-41$	M1A1 A1 (ag) 3	
(iv)	$\left(\sum\alpha\right)\left(\sum\alpha^2\right)=\sum\alpha^3+\sum\alpha^2\beta$ $\sum\alpha^3=(-4)(3)-(-41)$ $=29$	M1A1 M1 A1 cao 4	*or* $\left(\sum\alpha\right)^3=\sum\alpha^3+3\sum\alpha^2\beta+6\alpha\beta\gamma$ *or* $2\sum\alpha^3+8\sum\alpha^2+13\sum\alpha-30=0$ *Dependent on previous M1*
(v)	$\sum\frac{\beta+\gamma}{\alpha}=\frac{\sum\alpha^2\beta}{\alpha\beta\gamma}\ \left(=-\frac{41}{5}\right)$ $\sum\left(\frac{\gamma+\alpha}{\beta}\right)\left(\frac{\alpha+\beta}{\gamma}\right)=\frac{\sum\alpha^3+\sum\alpha^2\beta+3\alpha\beta\gamma}{\alpha\beta\gamma}$ $=\frac{3}{5}$ $\left(\frac{\beta+\gamma}{\alpha}\right)\left(\frac{\gamma+\alpha}{\beta}\right)\left(\frac{\alpha+\beta}{\gamma}\right)=\frac{\sum\alpha^2\beta+2\alpha\beta\gamma}{\alpha\beta\gamma}$ $=-\frac{31}{5}$ Equation is $y^3+\frac{41}{5}y^2+\frac{3}{5}y+\frac{31}{5}=0$ $5y^3+41y^2+3y+31=0$	B1 M1A2 M1A1 M1 A1 cao 8	 Give A1 if one error *Dependent on M1M1 above* A0 if ‘ = 0 ’ is missing
	OR Let $y=\frac{-4-x}{x},\ x=\frac{-4}{y+1}$ M1A1 $2\left(\frac{-4}{y+1}\right)^3+8\left(\frac{-4}{y+1}\right)^2+13\left(\frac{-4}{y+1}\right)-10=0$ M2 $-128+128(y+1)-52(y+1)^2-10(y+1)^3=0$ M2 $10y^3+82y^2+6y+62=0$ A2		 Give A1 if just one error

2 (a)	$\cos 4\theta + \mathrm{j}\sin 4\theta = (\cos\theta + \mathrm{j}\sin\theta)^4$ $= \cos^4\theta + 4\cos^3\theta(\mathrm{j}\sin\theta) + 6\cos^2\theta(\mathrm{j}\sin\theta)^2 + \dots$ $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ $\tan 4\theta = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$ $= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$	M1 M1A1 A1 A1 cao 5	Must have binomial coefficients
(b)(i)	$z - \dfrac{1}{z} = 2\mathrm{j}\sin\theta,\quad z + \dfrac{1}{z} = 2\cos\theta$ $-64\sin^2\theta\cos^4\theta = \left(z - \dfrac{1}{z}\right)^2\left(z + \dfrac{1}{z}\right)^4$ $= z^6 + 2z^4 - z^2 - 4 - \dfrac{1}{z^2} + \dfrac{2}{z^4} + \dfrac{1}{z^6}$ $= \left(z^6 + \dfrac{1}{z^6}\right) + 2\left(z^4 + \dfrac{1}{z^4}\right) - \left(z^2 + \dfrac{1}{z^2}\right) - 4$ $= 2\cos 6\theta + 2(2\cos 4\theta) - 2\cos 2\theta - 4$ $\sin^2\theta\cos^4\theta = \frac{1}{16} + \frac{1}{32}\cos 2\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta$	B1B1 B1 M1A1 M2 A1 (ag) 8	May be implied For $-64\sin^2\theta\cos^4\theta$ If terms kept separate, award for $\cos 2\theta\cos 4\theta = \frac{1}{2}(\cos 6\theta + \cos 2\theta)$ Give M1 for one cos term (or if 2's omitted)
(ii)	$\int_0^1 \dfrac{x^2}{(1+x^2)^4}\mathrm{d}x = \int_0^{\frac{\pi}{4}} \dfrac{\tan^2\theta}{(\sec^2\theta)^4}\sec^2\theta\,\mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \sin^2\theta\cos^4\theta\,\mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \left(\frac{1}{16} + \frac{1}{32}\cos 2\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta\right)\mathrm{d}\theta$ $= \left[\frac{1}{16}\theta + \frac{1}{64}\sin 2\theta - \frac{1}{64}\sin 4\theta - \frac{1}{192}\sin 6\theta\right]_0^{\frac{\pi}{4}}$ $= \frac{\pi}{64} + \frac{1}{64} + \frac{1}{192}$ $= \frac{\pi}{64} + \frac{1}{48}$	M1 A1 A1 M1A1 M1 A1 (ag) 7	Substitution and $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$ Limits not required Limits required (may be implied by later work)

3 (i)	$\frac{1}{2}(e^x + e^{-x}) = k$ $e^{2x} - 2ke^x + 1 = 0$ $e^x = \dfrac{2k \pm \sqrt{4k^2 - 4}}{2} \quad (= k \pm \sqrt{k^2 - 1})$	M1 M1 A1		($\pm$ not required)
	OR $\sinh x = \sqrt{k^2 - 1}$ (when $x > 0$) M1 $k + \sqrt{k^2 - 1} = \cosh x + \sinh x = e^x$ M1A1			
	OR $\dfrac{d}{dk}\ln(k + \sqrt{k^2 - 1}) = \ldots = \dfrac{1}{\sqrt{k^2 - 1}}$ B2 $\ln(k + \sqrt{k^2 - 1}) = \cosh^{-1} k + C$ When $k = 1$, $0 = 0 + C$, so $C = 0$ B1			
	$(k - \sqrt{k^2 - 1})(k + \sqrt{k^2 - 1}) = k^2 - (k^2 - 1) = 1$ so $k - \sqrt{k^2 - 1} = \dfrac{1}{k + \sqrt{k^2 - 1}}$ $x = \ln(k + \sqrt{k^2 - 1})$ or $x = \ln\left(\dfrac{1}{k + \sqrt{k^2 - 1}}\right)$ $x = \pm\ln(k + \sqrt{k^2 - 1})$	M1 A1 (ag)	5	*or* since $\cosh(-x) = \cosh x$, if $x = \lambda$ is one solution, the other must be $x = -\lambda$ *or* (when $x < 0$), $\sinh x = -\sqrt{k^2 - 1}$, $k + \sqrt{k^2 - 1} = \cosh x - \sinh x = e^{-x}$
(ii)	$2(\cosh^2 x - 1) - 5\cosh x = 10$ $2\cosh^2 x - 5\cosh x - 12 = 0$ $(\cosh x - 4)(2\cosh x + 3) = 0$ $\cosh x = 4$ $x = \pm\ln(4 + \sqrt{15})$	M1 M1 A1 A1 cao	4	*Dependent on previous M1* *Ignore* $\cosh x = -\frac{3}{2}$ *if stated* *or* $x = \ln(4 \pm \sqrt{15})$ Give A0 if any other solutions stated
(iii)	$f'(x) = 4\sinh x\cosh x - 5\sinh x$ $= \sinh x(4\cosh x - 5)$ $f'(x) = 0$ when $\sinh x = 0$, $\cosh x = \frac{5}{4}$ $x = 0,\ x = \pm\ln 2$ Stationary points $(0, -5)$ $\left(\ln 2, -\frac{41}{8}\right)$ $\left(-\ln 2, -\frac{41}{8}\right)$	M1A1 M1 A1 A2	6	One term correct is sufficient for M1 *Accept* 0.69 *or* $\cosh^{-1}\frac{5}{4}$ *for* x *but* $y = -5.125$ *must be exact* Give A1 for one correct
(iv)	$\int_0^{\ln 10} f(x)\,dx = \int_0^{\ln 10}(\cosh 2x - 1 - 5\cosh x)\,dx$ $= \left[\frac{1}{2}\sinh 2x - x - 5\sinh x\right]_0^{\ln 10}$ $= \frac{1}{2}\sinh(2\ln 10) - \ln 10 - 5\sinh(\ln 10)$ $= \frac{1}{4}\left(100 - \frac{1}{100}\right) - \ln 10 - \frac{5}{2}\left(10 - \frac{1}{10}\right)$ $= \frac{99}{400} - \ln 10$	M1 M1A1 M1 A1 (ag)	5	Exact evaluation of $\sinh(\ln 10)$ *or* $\sinh(2\ln 10)$ *Do not accept (e.g.)* $\sinh(\ln 10) = 4.95$ *without any working*

4 (i)	$\frac{dy}{dx} = \frac{-\frac{c}{p^2}}{cp} = -\frac{1}{p^2}$ Tgt is $y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$ $\left(\text{i.e. } y = -\frac{x}{p^2} + \frac{2c}{p}\right)$	M1A1 M1A1 4	or $\frac{dy}{dx} = -\frac{c^2}{x^2}$
(ii)	A is $(2cp,\ 0)$, B is $\left(0, \frac{2c}{p}\right)$ Midpt of AB is $\left(\frac{2cp+0}{2}, \frac{0+\frac{2c}{p}}{2}\right)$, which is P	M1 M1A1 3	Or $\overrightarrow{AP} = \begin{pmatrix} -cp \\ \frac{c}{p} \end{pmatrix}$, $\overrightarrow{PB} = \begin{pmatrix} -cp \\ \frac{c}{p} \end{pmatrix}$
(iii)	C is $(2cq,\ 0)$, D is $\left(0, \frac{2c}{q}\right)$ Gradient of AD is $\frac{\frac{2c}{q}}{-2cp} = -\frac{1}{pq}$ Gradient of CB is $\frac{\frac{2c}{p}}{-2cq} = -\frac{1}{pq}$ Gradient of PQ is $\frac{\frac{c}{q} - \frac{c}{p}}{cq - cp} = \frac{cp - cq}{pq(cq - cp)} = -\frac{1}{pq}$ Hence AD, CB, PQ are parallel	M1 M1 B1 A1 4	
(iv)		B1 B1	Hyperbola with branches in first and third quadrants Foci F and G in approx correct positions. *Dependent on previous B1*
	$\lvert PF - PG \rvert$ is constant for all points on hyperbola, $\lvert PF - PG \rvert = RS$ with $R(c, c)$, $S(-c, -c)$ $= 2c\sqrt{2}$	M2 M1 A1	
	OR $PF = \sqrt{\left(cp - c\sqrt{2}\right)^2 + \left(\frac{c}{p} - c\sqrt{2}\right)^2}$ M1 $PF = \left\lvert c\left(p + \frac{1}{p} - \sqrt{2}\right)\right\rvert$, $PG = \left\lvert c\left(p + \frac{1}{p} + \sqrt{2}\right)\right\rvert$ M2 $\lvert PF = PG \rvert = 2c\sqrt{2}$ A1		<u>and</u> $PG = \sqrt{\left(cp + c\sqrt{2}\right)^2 + \left(\frac{c}{p} + c\sqrt{2}\right)^2}$ *Modulus not required*
	Bisector of angle FPG is the tangent at P Gradient is $-\frac{1}{p^2}$	M2 A1 9	

Examiner’s Report

LEGACY MODULES

Pure Mathematics 5 (5505)

General Comments

There were many excellent scripts, with about 15% of candidates scoring 50 marks or more (out of 60). However, very many candidates found it to be a difficult paper, and about 40% scored 30 marks or less. The great majority of candidates chose questions 1, 2 and 3.

Comments on Individual Questions

Question 1 (Roots of a cubic equation)

This question was attempted by almost every candidate, and was extremely well answered, with half of the attempts scoring 18 marks or more (out of 20). The principles were well understood and the algebra was handled competently.

(i) $-4,\ \frac{13}{2},\ 5$; (ii) 3; (iv) 29; (v) $5y^3+41y^2+3y+31=0$.

Question 2 (Complex numbers)

The average mark on this question was about 12. In part (a), the techniques for finding $\cos 4\theta$ and $\sin 4\theta$ were well understood, but many candidates were unable to use these to express $\tan 4\theta$ in terms of $\tan\theta$. In part (b)(i), most candidates knew how to proceed, and a good proportion carried out the work accurately. In the final part, a very large number failed to transform it to the integral of $\sin^2\theta\cos^4\theta$, and so were unable to make further progress.

(a) $\dfrac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$.

Question 3 (Hyperbolic functions)

This was the worst answered question, with an average mark of about 9. In part (i), the derivation of the logarithmic form was well understood, but very few justified the ± sign convincingly. In part (ii), very many candidates wrote the equation in terms of exponentials and were unable to deal with the resulting quartic equation. Part (iii) was reasonably well answered, although few obtained all three stationary points. There was a fair amount of good work in the final integration.

(ii) $x=\pm\ln(4+\sqrt{15})$; (iii) $(0,-5)$, $\left(\ln 2,\ -\frac{41}{8}\right)$, $\left(-\ln 2,\ -\frac{41}{8}\right)$.

Question 4 (Rectangular hyperbola)

This question was attempted by less than a quarter of the candidates. The first three parts were reasonably well answered. In part (iv), the graph was usually sketched correctly, but the final parts (*A*) and (*B*) were often omitted. These were intended to test knowledge of geometrical properties of the hyperbola, but only one or two candidates gave the correct value for $|\mathrm{PF}-\mathrm{PG}|$, and just a few knew that the angle bisector was the tangent.

(i) $y=-\dfrac{x}{p^2}+\dfrac{2c}{p}$; (iv)(*A*) $2c\sqrt{2}$, (*B*) $-\dfrac{1}{p^2}$.